Journal of Statistical Physics, Vol. 108, Nos. 5/6, September 2002 (© 2002)

The Dissipative Scale of the Stochastics Navier–Stokes Equation: Regularization and Analyticity

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Received November 1, 2001; accepted May 8, 2002

We prove spatial analyticity for solutions of the stochastically forced Navier–Stokes equation, provided that the forcing is sufficiently smooth spatially. We also give estimates, which extend to the stationary regime, providing strong control of both of the expected rate of dissipation and fluctuations about this mean. Surprisingly, we could not obtain non-random estimates of the exponential decay rate of the spatial Fourier spectra.

KEY WORDS: Analyticity; Stochastic Navier–Stokes; dissipation rate; Gevrey class regularity; invariant measures.

1. INTRODUCTION

Consider the incompressible Navier–Stokes equations driven by a white in time stochastic forcing $f(x, t)$:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \, \Delta \mathbf{u} + f \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0,$$

where ν is the viscosity and p the pressure. Though some of our statement can be translated to short time statements for 3D flows, we are mainly concerned here with statements about the long time behavior of 2D flows.

We begin by discussing a useful caricature of 2D flows which helps motivate our results. Imagine a forcing concentrated on the large scales.

Dedicated to David Ruelle and Yakov G. Sinai on the occasion of their 65th birthday.

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First notice that the dissipation in the system is due to the $\nu\, \Delta u$ term while the forcing keeps the systems from relaxing to zero, the globally attracting fix point of the unforced system. If the viscosity, and hence the dissipation, is small then the viscous term only acts on very small scales. In addition, if the forcing is restricted to large scales then there is a clear separation between the forced scales and the dissipating scales. This intermediate region is refereed to as the inertial range and is characterized by algebraic decay of the magnitude of the Fourier coefficients with increasing wave number. Over this inertial scale "energy" is essentially neither dissipated nor injected. The caricature states that because of the conservation of enstrophy and energy, vorticity is transported to small scales to be dissipated while energy is transported upscale to compensate for this flux to small scales. For a more complete exposition see refs. 1 and 2. This paper concentrates on the two dimensional periodic setting. The exact features of the story differ in other settings such as the whole space or a domain with boundary. However, the properties of the dissipation rate are likely similar.

This note is concerned with characterizing the scale at which the dissipation starts. In contrast to the inertial scales, in the dissipative scales the size of the $\mathbf{k}$th Fourier mode decays exponentially as the wave number $\mathbf{k}$ increases. As such, it is the character of the dissipative scales which ensures the spatial analyticity of the velocity field. In characterizing the dissipative scale rigorously there are at least two, often opposing, points of view. One can try to make the bounds as sharp as possible so that they give the best reflection of the "Truth." On the other hand, one can try to maintain the general nature of the estimates while giving strong analytic control over the properties of various quantities in the estimate. Since we see our rigorous estimates mainly as a tool in other rigorous investigations, we take the second route. In particular, we do not claim that our estimates correctly capture the scaling of the dissipation rate relative to the viscosity or the structure of the forcing. If fact, our introduction of γ in equation (10) produces estimates which scale inferior to those obtained by preceding directly from (8). However a straightforward assault on this equations currently eludes the author and hence the simpler treatment including the γ is presented here.

The core ideas of this paper date back to refs. 3 and 4. The specific structure of the argument is similar to that in ref. 5. One of the points of this note it that this structure is well suited to the stochastic setting.

It is extremely fitting that this note is dedicated to Yakov Sinai. It was in collaboration with him that the author first explored the issues raised in this paper. In fact a number of the theorems in this paper are similar in statement to those proved in ref. 6 which was written under his supervision. It is also with pleasure that I also dedicate this paper to Professor Ruelle.

I have learned much from his papers and expect to continue doing so for a long time to come.

As mentioned, the results in this note are an improvement over very similar results in ref. 6 obtained by different techniques. In particular, this work does not separate off the linear part of the equation. This splitting lead to weaker control of the fluctuations in the previous work. More comments on the relationship between the various results can be found after the statement of the main theorem and in the conclusion. A different take on similar questions was presented in ref. 7. There the control was less explicit and strong, but the scaling relative to the parameters much better.

2. THE SETUP AND MAIN RESULTS

It is convenient to rewrite (1) as an Itô equation on the space of divergence free vector fields, thereby eliminating the need for the condition $\nabla\cdot\mathbf{u}=0$. In addition, we restrict ourself to the 2D torus $\mathbb{T}^2$. This makes direct calculation in the Fourier basis possible.

Letting P_{div} denote the projection onto the space of divergence free vector fields, we will work on the Sobolev spaces

$$\mathbb{H}^r=\Big\{\mathbf{u}=(u^{(1)},u^{(2)})=\sum_{\mathbf{k}\in 2\pi\mathbb{Z}^2}\mathbf{u}_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}\text{ where }\mathbf{u}_\mathbf{k}=(u_k^{(1)},u_k^{(2)}),$$

$$\mathbf{u}_0=0,\ \mathbf{k}\cdot\mathbf{u}_\mathbf{k}=0,\ \sum_\mathbf{k}|\mathbf{k}|^{2r}|\mathbf{u}_\mathbf{k}|^2<\infty\Big\}.$$

We will write $\mathbb{L}^2$ for $\mathbb{H}^0$. Projecting (1) onto $\mathbb{L}^2$ produces

$$\begin{aligned} d\mathbf{u}(\mathbf{x},t,\omega)+\nu\Lambda^2\mathbf{u}\,dt&=B(\mathbf{u},\mathbf{u})\,dt+d\mathbf{W}(\mathbf{x},t,\omega)\\ \mathbf{u}(\mathbf{x},0)&=(u^{(1)}(\mathbf{x},0),u^{(2)}(\mathbf{x},0))\in\mathbb{L}^2\end{aligned}\tag{2}$$

where $\Lambda=\sqrt{P_{\text{div}}(-\Delta)}$, $B(\mathbf{u},\mathbf{v})=P_{\text{div}}(u\cdot v)$. $\mathbf{W}(x,t,\omega)$ is a white in time random field defined by

$$\mathbf{W}(x,t,\omega)=\sum_{\mathbf{k}\in 2\pi\mathbb{Z}^2}\boldsymbol{\sigma}_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}\omega_\mathbf{k}(t)$$

with $\omega=(\omega_\mathbf{k})_\mathbf{k}$ a collection of standard i.i.d Wiener processes on a probability space $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P},\theta_t)$. We associate Ω with the canonical space generated by all of the $d\omega_k(t)$. $\mathscr{F}$ and $\mathscr{F}_t$ are respectively the associated global σ-algebra and filtration generated by $\mathbf{W}(t)$. Lastly, θ_t is the shift on Ω defined by $\theta_t\,d\omega_k(s)=d\omega_k(s+t)$. The $\boldsymbol{\sigma}_\mathbf{k}=(\sigma_\mathbf{k}^{(1)},\sigma_\mathbf{k}^{(2)})\in\mathbb{C}\times\mathbb{C}$ set the amplitude of the forcing. They are chosen so that $P_{\text{div}}\boldsymbol{\sigma}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = 0$ to make the forcing divergence free. To enforce the reality of the vector field, we assume $\omega_{-\mathbf{k}}(t) = \omega_{\mathbf{k}}(t)$ and $\boldsymbol{\sigma}_{\mathbf{k}} = \boldsymbol{\sigma}_{-\mathbf{k}}$. Lastly we make the following standing assumption.

Assumption A1 (Standing Assumption). There exist fixed positive constants β_0 and C_0 so that

$$|\boldsymbol{\sigma}_{\mathbf{k}}| \leqslant C_0 e^{-\beta_0 |\mathbf{k}|}. \tag{3}$$

With this standing assumption it is reasonable to expect that the velocity field $\mathbf{u}$ is analytic in space. This is the simplest formulation of our main theorem. More precisely, we have the following:

Theorem 1. If $\mathbf{u}(x, t, \omega) = \sum_{\mathbf{k}} \mathbf{u}_{\mathbf{k}}(t, \omega)\, e^{i\mathbf{k}\cdot\mathbf{x}}$ solves (2) and $|\mathbf{u}(0)|_{\mathbb{H}^r} < \infty$ for some $r > \frac{5}{2}$ then there exist almost surely finite stochastic processes $\tau(t, \omega)$ and $h(t, \omega)$ defined on $[0, \infty)$ such that

- τ and h are continuous in time.
- $\tau(t) > 0$ for $t > 0$.
- $|\mathbf{u}_{\mathbf{k}}(t)| \leqslant h(t) \exp(-|\mathbf{k}|\ \tau(t))$ almost surely.
- τ and h have moments which are uniformly bounded in time.
- $\tau(t)$ and $h(t)$ satisfy the simple differential equations given in (10) and (9) respectively.

The requirement that $|\mathbf{u}(0)|_{\mathbb{H}^r}$ is finite is not troublesome as the estimates in Appendixes B and D show that the higher Sobolev norms are finite at any moment of time after the initial one. Hence the estimates can be started from there.

As stated, the first parts of Theorem 1 are no different than versions contained in ref. 6. The various versions contained there differ in the properties of the τ and h produced. In particular the control over the moments was not as strong. In most ways the proofs presented in this note are an improvement over those in ref. 6. (To be precise, the statements in ref. 6 require slightly better than exponential decay of the forcing however the techniques of some of the proofs are quite similar to the ones exposed here and could be modified to need only exponential decay.) Furthermore, the proof in this text is more straightforward.

In Section 4, we prove Theorem 1 and develop many more characteristics of τ and h. In particular we give control on the size of the fluctuations of τ and h. In Section 5, we examine the implications of the previous analysis on the support of any invariant measure. We show that with high probability any invariant measure is concentrated on functions with a certain decay rate or better (see Lemma 5.3). Though our goal was to obtain strong rigorous analytic control, we also give some information about how the τ and h scale with the parameters. In Section 7 we show that asymptotically in time

$$\mathbb{E}\tau \geqslant C \frac{\nu^{\frac{3}{2}}}{C_0 + C_1 \frac{\nu^{\frac{3}{2}}}{\beta_0} + \sqrt{\mathbb{E}\,|\Lambda^r \mathbf{u}|_{\mathbb{L}^2}^{3}}}.$$

Here the constants C, C_1 and C_2 are independent of the viscosity and the forcing but do depend on the domain (and hence are not unitless). If one is interested in the scaling as $\nu \to 0$ one obtains

$$\mathbb{E}\tau \geqslant C \frac{\nu^{\frac{3}{2}}}{\sqrt{\mathbb{E}\,|\Lambda^r \mathbf{u}|_{\mathbb{L}^2}^{3}}}$$

This scaling is certainly not optimal. Similarly, we see from Propositions 4.3 or 4.4, that in the stationary regime

$$\mathbb{E}h \leqslant C \left(\frac{\mathbb{E}\,|\Lambda^r \mathbf{u}|_{\mathbb{L}^2}^{3}}{\nu} + C_0 \right).$$

In Section 7, we discuss in more detail the scaling as $\nu \to 0$. However we emphasis, that the point here was not to obtain estimates which scale optimally. Rather we have endeavored to obtain estimates which have easily controlled fluctuations. This is realized by the relatively straight forward form of Eqs. (9) and (10).

3. ANALYTICITY AND GEVREY CLASS REGULARITY

The basic ideas of our technique date back at least to ref. 3. We introduce the operator $e^{\tau\Lambda}$ which is defined by the Fourier multiplier $e^{\tau|\mathbf{k}|}$. Since $|e^{\tau\Lambda}\mathbf{u}|_{\mathbb{L}^2}^2 = \sum e^{2\tau|\mathbf{k}|}\,|\mathbf{u}_{\mathbf{k}}|^2$, if we succeed in proving that $|e^{\tau\Lambda}\mathbf{u}|_{\mathbb{L}^2} < C$ then we can conclude that $|\mathbf{u}_{\mathbf{k}}| < Ce^{-\tau|k|}$.

We now give a brief account of the concept of Gevrey class regularity as it pertains to our problem. For a more complete account see refs. 5 and 8.

For $s > 0$, we define the Gevrey Class $\mathbb{G}^s$ by the $\mathbf{u} \in C^\infty \times C^\infty$ such that there exists a $\rho > 0$ and a $C < \infty$ so that for all $\mathbf{x} \in \mathbb{T}^2$ and multindex $\beta \in \mathbb{N}^2$ one has

$$\left| \frac{\partial^{|\beta|}\mathbf{u}(\mathbf{x})}{\partial x_1^{\beta_1}\, \partial x_2^{\beta_2}} \right| \leqslant C \left(\frac{\beta!}{\rho^{|\beta|}} \right)^s$$

where $\beta = (\beta_1, \beta_2)$, $\beta! = \beta_1!\,\beta_2!$ and $|\beta| = |\beta_1| + |\beta_2|$. It is straight forward to verify that $\mathbb{G}^s$ is closed under multiplication, differentiation, and composition. Also $\mathbb{G}^{s_1} \subset \mathbb{G}^{s_2}$, if $s_1 < s_2$.

In light of the definition, it is not surprising that we can connect the above concepts with the idea of smoothness as characterized by Sobolev norms. In ref. 5, one finds the following result:

Fix $s > 0$ and $r \geqslant 0$. Then $\mathbf{u} \in \mathbb{G}^s$ if and only if there exists $\rho > 0$ and $C < \infty$, possibly depending on r, s, and $\mathbf{u}$, such that for every $n \in \mathbb{N}$

$$|\nabla^n \mathbf{u}|_{\mathbb{H}^r} \leqslant C \left(\frac{n!}{\rho^n} \right)^s.$$

Finally, we connect back with the operator $e^{\tau \Lambda}$ which was introduced at the start of the section. If $\mathscr{D}(e^{\tau \Lambda^{1/s}} : \mathbb{H}^r) = \{\mathbf{u} \in \mathbb{H}^r : |e^{\tau \Lambda^{1/s}} \mathbf{u}|_{\mathbb{H}^r} < \infty\}$ then, again from ref. 5, $\mathbb{G}^s = \bigcup_{\tau > 0} \mathscr{D}(e^{\tau \Lambda^{1/s}} : \mathbb{H}^r)$ for any $r \geqslant 0$ and $s > 0$.

Hence the operator $e^{\tau \Lambda^{1/s}}$ can be used to prove membership in $\mathbb{G}^s$. We are mainly interested in $s = 1$ as it corresponds to real analytic functions.

In the following, we will not work with $e^{\tau \Lambda^{1/s}}$ but rather $\Lambda^r e^{\tau \Lambda^{1/s}}$ for some $r > \frac{5}{2}$. This is so that our spaces are Banach algebras. In particular in refs. 5 and 9, it is proved that if $s \geqslant 1$, $\tau > 0$, $r > \frac{d}{2}$ then $\mathbb{H}^r$ is an Banach algebra and

$$|\Lambda^r e^{\tau \Lambda^{1/s}} (\mathbf{u}\mathbf{v})|_{\mathbb{H}^r} \leqslant C(r, d)\, |\Lambda^r e^{\tau \Lambda^{1/s}} \mathbf{u}|_{\mathbb{H}^r}\, |\Lambda^r e^{\tau \Lambda^{1/s}} \mathbf{v}|_{\mathbb{H}^r}.$$

4. THE HEART OF THE ANALYSIS

Throughout this section, we will work with an arbitrary Galerkin approximation $\mathbf{u}^{(N)}$ defined by projecting (2) onto the Fourier modes with wave number $|\mathbf{k}| \leqslant N$. $\mathbf{u}^{(N)}(\mathbf{x}, t) = \sum_{|\mathbf{k}| \leqslant N} \mathbf{u}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}$ solves

$$d\mathbf{u}^{(N)} = [-\nu \Lambda^2 \mathbf{u}^{(N)} + P_N B(\mathbf{u}^{(N)}, \mathbf{u}^{(N)})]\, dt + \sum_{|\mathbf{k}| \leqslant N} \boldsymbol{\sigma}_k\, d\omega_k(t). \tag{4}$$

Working with the Galerkin approximation removes the question of the fitness of terms like $|\Lambda^r e^{\tau \Lambda} \mathbf{u}|_{\mathbb{L}^2}$ in intermediate steps. Initially we must use a finite Galerkin approximations in statements like (5) below, as we do not know if they are finite for the full solution. However, from the start all of the estimates we obtain are independent of the choice of N used to define Galerkin approximations. We need only use the Galerkin approximation to make the manipulations possible. Since all of the estimates are uniform in the order of the Galerkin approximation, we pass to the limit trivially at the very end of our argument. As nothing will depend on N, we suppress it from our notation in the name of readability.

The presentation here closely parallels that in ref. 6 which in turn closely parallels the presentation in ref. 5. Relative to ref. 6 there are some important differences which give estimates which are more uniform in time.

Recalling that by definition

$$|\Lambda^r e^{\tau\Lambda}\mathbf{u}|^2_{\mathbb{L}^2} = \sum_{\mathbf{k}} |\mathbf{k}|^{2r}\, |\mathbf{u}_{\mathbf{k}}|^2\, e^{2\tau\,|\mathbf{k}|} \tag{5}$$

and allowing the possibility that τ depends on time (but has finite first variation in time), Itô's formula implies that

$$\begin{aligned} d\,|\Lambda^r e^{\tau\Lambda}\mathbf{u}|^2_{\mathbb{L}^2} &= \Bigg[-2\nu\,|\Lambda^{r+1}e^{\tau\Lambda}\mathbf{u}|^2_{\mathbb{L}^2} + 2\langle P_N \Lambda^r e^{\tau\Lambda} B(\mathbf{u},\mathbf{u}),\, \Lambda^r e^{\tau\Lambda}\mathbf{u}\rangle_{\mathbb{L}^2} \\ &\quad + 2\frac{d\tau}{dt}\,|\Lambda^{r+\frac{1}{2}}e^{\tau\Lambda}\mathbf{u}|^2_{\mathbb{L}^2} + \sum_{|\mathbf{k}|\leqslant N} |\mathbf{k}|^{2r}\, e^{2\tau\,|\mathbf{k}|}\,|\boldsymbol{\sigma}_k|^2\Bigg] dt + 2\langle \Lambda^{2r}e^{2\tau\Lambda}\mathbf{u},\, d\mathbf{W}(t)\rangle_{\mathbb{L}^2}. \end{aligned} \tag{6}$$

Next we use a lemma proved in ref. 5 to estimate the nonlinear term.

Lemma 4.1. Let $\mathbf{u} \in \mathscr{D}(\Lambda^r e^{\tau\Lambda})$ with $r > \frac{d}{2} + \frac{3}{2}$, such that $\mathbf{u}$ is mean zero and $\nabla\cdot\mathbf{u} = 0$ then

$$|\langle \Lambda^r e^{\tau\Lambda} B(\mathbf{u},\mathbf{u}),\, \Lambda^r e^{\tau\Lambda}\mathbf{u}\rangle_{\mathbb{L}^2}| \leqslant C\,|\Lambda^r\mathbf{u}|^3_{\mathbb{L}^2} + \tau C\,|\Lambda^r e^{\tau\Lambda}\mathbf{u}|_{\mathbb{L}^2}\,|\Lambda^{r+\frac{1}{2}}e^{\tau\Lambda}\mathbf{u}|^2_{\mathbb{L}^2}$$

for some $C > 0$ independent of τ.

Writing Eq. (4) in integral form, we use this lemma to produce

$$\begin{aligned} &|\Lambda^r e^{\tau(t)\,\Lambda}\mathbf{u}(t)|^2_{\mathbb{L}^2} \\ &\quad\leqslant |\Lambda^r e^{\tau(0)\,\Lambda}\mathbf{u}(0)|^2_{\mathbb{L}^2} + \int_0^t 2\langle \Lambda^{2r}e^{2\tau(s)\,\Lambda}\mathbf{u}(s),\, d\mathbf{W}(s)\rangle_{\mathbb{L}^2} \\ &\qquad + \int_0^t 2\Bigg[-\nu\,|\Lambda^{r+1}e^{\tau(s)\,\Lambda}\mathbf{u}(s)|^2_{\mathbb{L}^2} + \sum_{\mathbf{k}} |\mathbf{k}|^{2r}\, e^{2\tau(s)\,|k|}\,|\boldsymbol{\sigma}_{\mathbf{k}}|^2 + C\,|\Lambda^r\mathbf{u}(s)|^3_{\mathbb{L}^2} \\ &\qquad + \tau C\,|\Lambda^r e^{\tau(s)\,\Lambda}\mathbf{u}(s)|_{\mathbb{L}^2}\,|\Lambda^{r+\frac{1}{2}}e^{\tau(s)\,\Lambda}\mathbf{u}(s)|^2_{\mathbb{L}^2} + \frac{d\tau}{dt}\,|\Lambda^{r+\frac{1}{2}}e^{\tau(s)\,\Lambda}\mathbf{u}(s)|^2_{\mathbb{L}^2}\Bigg] ds. \end{aligned}$$

As $\mathbf{u}$ is mean zero, the Poincaré inequality implies that

$$|\Lambda^{r+1} e^{\tau\Lambda}\mathbf{u}|_{\mathbb{L}^2}^2 \geqslant 2\pi\, |\Lambda^{r+\frac{1}{2}} e^{\tau\Lambda}\mathbf{u}|_{\mathbb{L}^2}^2 \geqslant 4\pi^2\, |\Lambda^{r} e^{\tau\Lambda}\mathbf{u}|_{\mathbb{L}^2}^2$$

and hence,

$$\begin{aligned}
&|\Lambda^r e^{\tau(t)\,\Lambda}\mathbf{u}(t)|_{\mathbb{L}^2}^2\\
&\leqslant |\Lambda^r e^{\tau(0)\,\Lambda}\mathbf{u}(0)|_{\mathbb{L}^2}^2 + \int_0^t 2\left[-\pi\nu + \tau C\, |\Lambda^r e^{\tau(s)\,\Lambda}\mathbf{u}(s)|_{\mathbb{L}^2} + \frac{d\tau}{dt}(s)\right] |\Lambda^{r+\frac{1}{2}} e^{\tau(s)\,\Lambda}\mathbf{u}(s)|_{\mathbb{L}^2}^2\, ds\\
&\quad + \int_0^t \left[2C\, |\Lambda^r\mathbf{u}(s)|_{\mathbb{L}^2}^3 - 4\nu\pi^2\, |\Lambda^r e^{\tau(s)\,\Lambda}\mathbf{u}(s)|_{\mathbb{L}^2}^2 + \sum_{\mathbf{k}} |\mathbf{k}|^{2r}\, e^{2\tau\,|\mathbf{k}|}\, |\boldsymbol{\sigma}_{\mathbf{k}}|^2\right] ds\\
&\quad + \int_0^t 2\langle \Lambda^{2r} e^{2\tau(s)\,\Lambda}\mathbf{u}(s), d\mathbf{W}(s)\rangle_{\mathbb{L}^2}.
\end{aligned} \tag{7}$$

Up to now we have made no restrictions on the τ dynamics other than it should have finite first variation in time. If we chose the dynamics such that

$$\frac{d\tau}{dt} - \pi\nu + \tau C\, |\Lambda^r e^{\tau\Lambda}\mathbf{u}|_{\mathbb{L}^2} \leqslant 0 \tag{8}$$

then the first integral in (7) can be neglected. For the moment, we assume this can be done and see what the implications would be. Continuing under this assumption, we have

$$\begin{aligned}
|\Lambda^r e^{\tau(t)\,\Lambda}\mathbf{u}(t)|_{\mathbb{L}^2}^2 &\leqslant |\Lambda^r e^{\tau(0)\,\Lambda}\mathbf{u}(0)|_{\mathbb{L}^2}^2\\
&\quad + \int_0^t [2C\, |\Lambda^r\mathbf{u}(s)|_{\mathbb{L}^2}^3 - 4\nu\pi^2\, |\Lambda^r e^{\tau(s)\,\Lambda}\mathbf{u}(s)|_{\mathbb{L}^2}^2 + f_r(\tau)]\, ds + N(s)
\end{aligned}$$

where $f_r(\tau) = \sum_{\mathbf{k}} |\mathbf{k}|^{2r}\, e^{2\tau\,|\mathbf{k}|}\, |\boldsymbol{\sigma}_{\mathbf{k}}|^2$, $dN(s) = -2\pi^2\nu\, |\Lambda^r e^{\tau\Lambda}\mathbf{u}|_{\mathbb{L}^2}^2\, ds + dM(s)$ and $dM(s) = \langle \Lambda^{2r} e^{2\tau\Lambda}\mathbf{u}(s), d\mathbf{W}(s)\rangle_{\mathbb{L}^2}$. In light of the assumptions on the $\boldsymbol{\sigma}_{\mathbf{k}}$, it is straight forward to verify that $f_r(\tau)$ is strictly increasing in τ and finite as long as $\tau < \beta_0$. Let us further assume that we can pick τ dynamics which so that $\tau(t) \leqslant \beta_1 < \beta_0$ almost surely for all $t \geqslant 0$ and some fixed β_1. Under this assumption $f_r(\tau) \leqslant f_r^* = f_r(\beta_1)$ and hence if we define the stochastic process $h(t)$ by the Itô SDE

$$dh(t) = [-2\pi^2\nu h(t) + 2C\, |\Lambda^r\mathbf{u}|_{\mathbb{L}^2}^3 + f_r(\tau)]\, dt + dN(t) \tag{9}$$

then $|A^r e^{\tau(t)A}\mathbf{u}(t)|^2_{\mathbb{L}^2} \leqslant h(t)$ almost surely. All that remains is to specify the dynamics of τ. As long as (8) and $\tau(t) \leqslant \beta_1$ are satisfied, we are free to pick almost any dynamics we wish. For any fixed $\gamma \geqslant 0$, consider the dynamics

$$\frac{d\tau}{dt} + c\tau(\gamma + \sqrt{h}) = \pi\nu. \tag{10}$$

Begin by noticing that, because of the ν on the right hand side, $\tau(t) > 0$ as long as $\tau(0) \geqslant 0$. Second notice that because $\tau \geqslant 0$ and $h(t) \geqslant 0$ we have that

$$\frac{d\tau}{dt} - \pi\nu + C\tau\,|A^r e^{\tau A}\mathbf{u}|_{\mathbb{L}^2} \leqslant \frac{d\tau}{dt} - \pi\nu + C\tau\sqrt{h} \leqslant \frac{d\tau}{dt} - \pi\nu + C\tau(\gamma + \sqrt{h}).$$

We conclude that the choice of τ dynamics given by (10) satisfies the condition in (8). To satisfy the remaining condition on τ, we need a bound from above.

Proposition 4.2. If $\gamma > 0$ then,

$$\tau(t) \leqslant \tau(0)\exp\{-C\gamma t\} + \frac{\nu}{C\gamma}(1 - \exp\{-C\gamma t\})$$

$$\tau(t) \geqslant \tau(0)\exp\left\{-Ct\left[\gamma + \sqrt{\frac{1}{t}\int_0^t h(\xi)\,d\xi}\right]\right\}$$
$$+ \nu\int_0^t \exp\left\{-C(t-s)\left[\gamma + \sqrt{\frac{1}{t-s}\int_s^t h(\xi)\,d\xi}\right]\right\}$$

almost surely. And

$$\mathbb{E}\tau(t) \geqslant \tau(0)\exp\left\{-Ct\left[\gamma + \sqrt{\frac{1}{t}\int_0^t \mathbb{E}h(\xi)\,d\xi}\right]\right\}$$
$$+ \nu\int_0^t \exp\left\{-C(t-s)\left[\gamma + \sqrt{\frac{1}{t-s}\int_s^t \mathbb{E}h(\xi)\,d\xi}\right]\right\}$$

Proof. Both follow from integrating (10) using the variation of parameter formula. The first is obtained by neglecting the integral containing $\sqrt{h}$. The second follows from the fact that $\frac{1}{t}\int_0^t \sqrt{h} \leqslant \sqrt{\frac{1}{t}\int_0^t h}$ by Jensen's inequality. The third follows from the second by repeatedly using Jensen's inequality to move the expectation inward. ▮

From the first part of this proposition, it is clear that if $\gamma > \frac{\nu}{C\beta_1}$ then $\tau(t) \leqslant \beta_1$ almost surely. From now on we assume that this condition holds.

In light of the above discussion, we have proved the following theorem which implies Theorem 1.

Theorem 2. Assume that the $\boldsymbol{\sigma}_{\mathbf{k}}$ are such that the standing assumption (1) holds. Fix an $r > \frac{5}{2}$ and a $\gamma \geqslant \frac{\nu}{C\beta_1}$ where $\beta_1 < \beta_0$. If $\tau(0) \geqslant 0$ and $|\Lambda^r e^{\tau(0)\,\Lambda} \mathbf{u}(0)|_{\mathbb{L}^2} < \infty$ then Theorem 1 holds for τ and h defined by (10) and (9) respectively.

We now concentrate on extracting the properties of the system defined by (9) and (10) which will in turn imply statements about the regularity of the solution $\mathbf{u}(t)$ of the stochastic Navier–Stokes equation.

In practice, we will mainly be interested in the case when $\tau(0) = 0$ and hence $|\Lambda^r e^{\tau(0)\,\Lambda}\mathbf{u}|_{\mathbb{L}^2} = |\Lambda^r \mathbf{u}|_{\mathbb{L}^2}$. This assumptions simplifies some formuli which follow, but is not truly necessary.

Proposition 4.3. If $\tau(0) = 0$ then,

$$\mathbb{E}h(t) \leqslant \mathbb{E}h(0)\, e^{-2\pi\nu t} + \int_0^t 2Ce^{-2\pi\nu(t-s)} \mathbb{E}\, |\Lambda^r \mathbf{u}(s)|^3_{\mathbb{L}^2}\, ds + K_0(1 - e^{-2\pi\nu t})$$

where $K_0 = \frac{f_r^*}{2\pi\nu}$. Notice that since $\beta_1 < \beta_0$, f_r^* is $O(C_0)$ regardless of the other details of the $\boldsymbol{\sigma}_{\mathbf{k}}$'s. ($C_0$ is the constant in the standing assumption.) Hence K_0 is always $O(\frac{C_0}{\nu})$.

Proof. Conceptually the proof is straightforward. One simply integrates (9) and takes its expectation. Because all of the integrands in the Stiltjes integrals are positive one can exchange the expectation and the integral in time. We would be done if we could then claim that the expected value of integral against the semi-martingale dN is less than or equal to zero. Specifically we would like to claim that

$$\mathbb{E} \int_0^t e^{-2\pi\nu(t-s)}\, dN(s) \leqslant 0.$$

We see from the definition of N that this integral is made up of two terms. The first is the regular Stieltjes,

$$-\mathbb{E} \int_0^t e^{-2\pi\nu(t-s)}\, |\Lambda^r e^{\tau(s)\,\Lambda} \mathbf{u}(s)|^2_{\mathbb{L}^2}\, ds$$

which is clearly less than or equal zero and hence can be neglected. This leaves only the expected value of the Itô integral

$$e^{-2\pi\nu t}\mathbb{E}\tilde{M}(t) = e^{-2\pi\nu t}\mathbb{E}\int_0^t e^{2\pi\nu s}\langle \Lambda^{2r} e^{2\tau(s)\Lambda}\mathbf{u}(s), d\mathbf{W}(s)\rangle_{\mathbb{L}^2}.$$

This expected value is zero as long as we know that the expected value of the quadratic variation of $\tilde{M}(t)$ is finite. Unfortunately, the quadratic variation

$$[\tilde{M}, \tilde{M}] = \int_0^t e^{4\pi\nu s} \sum |\mathbf{k}|^{4r} |\boldsymbol{\sigma}_{\mathbf{k}}|^2 e^{4\tau|\mathbf{k}|} |\mathbf{u}_{\mathbf{k}}|^2 \leqslant \hat{f}_r(\beta_1) \int_0^t e^{4\pi\nu s} |\Lambda^r e^{\tau(s)\Lambda}\mathbf{u}(s)|^2_{\mathbb{L}^2}\, ds \tag{11}$$

and hence it is naturally bounded exactly by the term we are trying to control with $h(t)$. (Here $\hat{f}_r(\tau) = \max |\mathbf{k}|^{2n} e^{2\tau|\mathbf{k}|} |\boldsymbol{\sigma}_{\mathbf{k}}|^2$. This is finite because $\tau \leqslant \beta_1 < \beta_0$ almost surely.) Fortunately there is a standard way around this quandary. For any $U > 0$ we introduce the stopping time $T = \inf\{s\colon |\Lambda^r e^{\tau(s)\Lambda}\mathbf{u}(s)|^2_{\mathbb{L}^2} \geqslant U\}$ and the stopped process $u^T(t) = u(T\wedge t)$. Here $T\wedge t = \min(T, t)$. As long a $t < T$, u^T still satisfies (4). Because of the definition of the stopping time, (11) is clearly finite if u is replaced with u^T. Hence the proposition holds for a $h^T(s)$ defined by

$$dh^T(t) = [-2\pi^2\nu h^T(t) + 2C\,|\Lambda^r\mathbf{u}|^3_{\mathbb{L}^2} + f_r(\tau)]\, dt + dN^T(t) \tag{12}$$

where $dN^T(s) = -2\pi^2\nu\,|\Lambda^r e^{\tau\Lambda}\mathbf{u}|^2_{\mathbb{L}^2}\, ds + dM^T(s)$ and $dM^T(s) = \langle \Lambda^{2r} e^{2\tau\Lambda}\mathbf{u}^T, d\mathbf{W}(s)\rangle_{\mathbb{L}^2}$. Since $\mathbf{u}(t)$ is continuous in time and finite almost surely, $T\wedge t$ converges to t as the cut-off $U\to\infty$. Hence, $\mathbf{u}^T \to \mathbf{u}$ as $U\to\infty$. Since the bound obtained on h^T is independent of U, we can transfer the bound to h by taking the limit as $U\to\infty$. ▮

With Proposition 4.3 in hand, we know that the expectation of quadratic variation $[M, M](t)$ is finite. From this fact, one obtains strong control of $N(t)$. The key to this control is to observe that N has a particular form. $N - M$ has finite first variation and $d(N(t) - M(t)) \leqslant -\frac{A}{2}\, d[M, M](t)$ where $A = 2\nu/\hat{f}^*_r$. A standard exponential martingale estimate then implies that $\mathbb{P}\{\sup_{s\leqslant t} N(s) > b\} \leqslant e^{-bA}$. From this exponential control of the deviations many useful facts follow. For instance, it is straightforward to prove that $\lim_{t\to\infty} \frac{N(t)}{t} = 0$ almost surely (see Appendix A for an analogous calculation or ref. 10 for an example in just this setting). This coupled with the fact that $f_r(\tau) \geqslant 0$ produces the following proposition.

Proposition 4.4. With probability one

$$\frac{2C}{2\pi\nu}\lim_{t\to\infty}\frac{1}{t}\int_0^t |\Lambda^r\mathbf{u}|_{\mathbb{L}^2}^{\frac{3}{2}}\,ds \leqslant \lim_{t\to\infty}\frac{1}{t}\int_0^t h(s)\,ds \leqslant \frac{2C}{2\pi\nu}\lim_{t\to\infty}\frac{1}{t}\int_0^t |\Lambda^r\mathbf{u}|_{\mathbb{L}^2}^{\frac{3}{2}}\,ds + K_0 \tag{13}$$

where K_0 is the constant from Proposition 4.3.

This is already useful, but we can do better. Integrating up (9) produces

$$h(t) = h(0)\,e^{-2\pi^2\nu t} + 2C\int_0^t e^{-2\pi^2\nu(t-s)}\,|\Lambda^r\mathbf{u}(s)|_{\mathbb{L}^2}^{\frac{3}{2}}\,ds$$

$$+\frac{C_1}{2\pi^2\nu}(1-e^{-2\pi^2\nu t}) + \int_0^t e^{-2\pi^2\nu(t-s)}\,dN(s).$$

Hence $d[N(t)-M(t)] \leqslant -\frac{A}{2}\,d[M,M](t)$ almost surely. Thus setting $\tilde{N}(t) = -\frac{A}{2}[M,M](t)+M(t)$, we have that $\int_0^t e^{-2\pi^2\nu(t-s)}\,dN(s) \leqslant \int_0^t e^{-2\pi^2\nu(t-s)}\,d\tilde{N}(s)$ and Lemma A.1 implies

$$\tilde{N}(t) = -\frac{A}{2}[M,M](t)+M(t) \leqslant \frac{e^1\hat{f}_r^*}{2\nu}[K+2\log(2\pi^2\nu t+1)]$$

with at least probability $1-\frac{\pi^2}{6}e^{-K}$. Hence, if we define

$$g_{\exp}(t,K) = h(0)\,e^{-2\pi^2\nu t} + \frac{C}{\pi^2\nu}\,g_r(t,K)^{\frac{3}{2}}$$

$$+\frac{C_1}{2\pi^2\nu}(1-e^{-2\pi^2\nu t}) + \frac{e^1\hat{f}_r^*}{2\nu}[K+2\log(2\pi^2\nu t+1)]$$

where g_r is defined in Corollary D.3, then we have the following lemma.

Lemma 4.5. With at least probability $1-\frac{\pi^2}{2}e^{-K}$, $h(t) \leqslant g_{\exp}(t,K)$ for all $t \geqslant 0$.

Notice that at fixed time $g_{\exp}(t,K)$ grows like $K^{\frac{3}{2}(r+1)}$. And for fixed K it grows like $[2\log(\frac{\nu}{2}t+1)]^p$ in t for some $p \geqslant 1$.

5. IMPLICATIONS FOR INVARIANT MEASURES

We now wish to show that the estimates in the previous section and in the Appendix, imply that any invariant measure within a certain class must be quite regular. Recall that we defined $\mathscr{E}_n = \sum |\mathbf{k}|^{2n}\,|\boldsymbol{\sigma}_{\mathbf{k}}|^2$. The finiteness of the $\mathscr{E}_n$ and Assumption A.1 will be the main ways of characterizing the smoothness.

We are interested in deriving regularity statements characterizing any stationary measures which might exist. As the standard treatment of existence and uniqueness[11, 12] assumes that the initial data is in $\mathbb{L}^2$, we will assume that for any invariant probability measure μ under discussion there exists a $U \subset \mathbb{L}^2$ such that $\mu(U) = 1$. For the same reason, we always assume $\mathscr{E}_0 < \infty$. In this setting, the following lemma was proved in ref. 10.

Lemma 5.1. Consider any invariant probability measure, as described above.

(1) All energy moments are finite. In fact for any $p \geqslant 1$ there exist a constant $C_p < \infty$ such that

$$\int_{\mathbb{L}^2} |\mathbf{u}|_{\mathbb{L}^2}^{2p} \, d\mu(\mathbf{u}) < C_p.$$

In particular $C_1 = \frac{\mathscr{E}_0}{2\nu}$.

(2) The first moment of the enstrophy is finite. Therefore

$$\int_{\mathbb{L}^2} |\Lambda \mathbf{u}|_{\mathbb{L}^2}^2 \, d\mu(\mathbf{u}) = \frac{\mathscr{E}_0}{2\nu}.$$

In addition, if the forcing is such that $\mathscr{E}_1 < \infty$ then

$$\int_{\mathbb{L}^2} |\Lambda^2 \mathbf{u}|_{\mathbb{L}^2}^2 \, d\mu(\mathbf{u}) = \frac{\mathscr{E}_1}{2\nu} \qquad \text{and} \qquad \int_{\mathbb{L}^2} |\Lambda \mathbf{u}|_{\mathbb{L}^2}^{2p} \, d\mu(\mathbf{u}) < C_1(p) < \infty$$

for all $p \geqslant 1$.

We now expand on these ideas. Let μ be an invariant measure as above. Let $u(0)$ be the random variable obtained by starting at time $t_0 < 0$ with initial data distributed according to μ. Since μ is an invariant measure

$$\int \mathbf{1}_{|\Lambda \mathbf{u}(t_0)|_{\mathbb{L}^2}^2 > \frac{\mathscr{E}_1}{\nu} + K}(\mathbf{u}) \, d\mu(\mathbf{u}) = \int \mathbb{E} \mathbf{1}_{|\Lambda \mathbf{u}(0)|_{\mathbb{L}^2}^2 > \frac{\mathscr{E}_1}{\nu} + K}(\mathbf{u}) \, d\mu(\mathbf{u}).$$

Next notice that

$$\int \mathbb{E} \mathbf{1}_{|\Lambda \mathbf{u}(0)|_{\mathbb{L}^2}^2 > \frac{\mathscr{E}_1}{\nu} + K}(\mathbf{u}) \, d\mu(\mathbf{u}) \leqslant \mathbb{P}\left\{ |\Lambda \mathbf{u}(t_0)|_{\mathbb{L}^2}^2 \, e^{-\nu |t_0|} > \frac{K}{2} \right\}$$
$$+ \mathbb{P}\left\{ |\Lambda \mathbf{u}(0)|_{\mathbb{L}^2}^2 - |\Lambda \mathbf{u}(t_0)|_{\mathbb{L}^2}^2 \, e^{-\nu |t_0|} > \frac{\mathscr{E}_1}{\nu} + \frac{K}{2} \right\}.$$

Since $\int |A\mathbf{u}|_{\mathbb{L}^2}^2 \, d\mu(\mathbf{u}) < \infty$, we estimate the first probability by Chebechev's inequality producing

$$\mathbb{P}\left\{ |A\mathbf{u}(t_0)|_{\mathbb{L}^2}^2 \, e^{-\nu|t_0|} > \frac{K}{2} \right\} \leqslant \frac{2e^{-2\nu|t_0|}}{K} \int |A\mathbf{u}|_{\mathbb{L}^2}^2 \, d\mu(\mathbf{u}).$$

Using Lemma B.2, we know that the second probability decays exponentially like $ce^{-\nu\gamma' K}$ for some positive c and γ' which are independent of t_0 and ν. Taking $t_0 \to -\infty$, we obtain the following lemma:

Lemma 5.2. For some positive γ and C independent of ν and μ, $\int \exp(\gamma\nu \, |A\mathbf{u}|_{\mathbb{L}^2}^2) \, d\mu(\mathbf{u}) < C < \infty$.

In the same manner, we can obtain estimates implying that any invariant measure is supported on the space of analytic functions. We define $\mathscr{U}_{\tau,h} = \{\mathbf{u}\colon |\mathbf{u}_{\mathbf{k}}| < he^{-\tau|\mathbf{k}|} \text{ for all } \mathbf{k}\}$, and $\mathscr{U} = \bigcup_{\tau>0} \bigcup_{h>0} \mathscr{U}_{\tau,h}$. Proposition 4.2 ensures that $\mu(\mathscr{U}) = 1$. From the definition of τ it is easy to see that if one knows that $h(t) < h_*$ for some positive h_* and all $t \in [-T, 0]$ where $t_0 \ll -T < 0$ then we know that $\tau(0) > \frac{C}{\gamma + \sqrt{h_*}}$ where C is some positive constant depending on T, ν and γ, but not h_*. With analysis similar to that used above for the enstrophy, but based on the estimates from Lemma 4.5, gives the existence positive constants c and δ so that the $\mu(\bigcup_{\tau>0} \mathscr{U}_{\tau,h_*}) \geqslant 1 - ce^{-\delta h_*}$. Combining these two observations gives the following lemma.

Lemma 5.3. There exists positive constants c, C, and δ so that for any invariant measure $\mu(\mathscr{U}_{\frac{C}{\gamma+\sqrt{h_*}}, h_*}) > 1 - ce^{-\delta h_*}$.

6. IMPLICATIONS FOR QUESTIONS OF ERGODICITY

We close by considering the implications of Theorem 2 on questions of ergodicity. The first rigorous proofs of the ergodicity of Eq. (2) were produced by ref. 13 and then extended by ref. 14. These results required that the $\boldsymbol{\sigma}_{\mathbf{k}}$ have decay bounded below by some algebraic power of $|\mathbf{k}|$. Since we now see that the natural decay of the $|\mathbf{u}_{\mathbf{k}}|$ is exponential is $\mathbf{k}$, these results seem to relay on the noise to overwhelm the natural fine structure of the problem. This means that the noise set the relevant topology in the function space allowing one to write densities relative to a fixed reference measure. If in fact the true decay does fluctuate as τ does this means that the transition densities are note defined relative to a measure with a common fixed decay. This seems troublesome. In ref. 15 the question of the correct topology was sidestepped completely by working at low enough

Reynolds number for the dynamics to be globally contracting. Finally in ref. 10, a technique was developed which worked for any Reynolds number and respected the fine scale structure of the nonlinearity with the mild imposition of an "effective" ellipticity condition (see ref. 16). This method also side stepped the question of topology by reducing the dynamics to an equation with memory on $\mathbb{R}^n$. Similar ideas were used to obtain similar ergodicity results in ref. 17.

7. IMPLICATIONS FOR SCALING WITH ν

As stated at the onset, our goal was not to obtain estimates which scaled faithfully as ν and the forcing strength were varied. Nonetheless it is worth observing how the estimates we have obtained scale. Even with in the framework, we have made some choices which make our estimates less than optimal. In particular, our introduction of γ in (10) was an expedient choice to keep τ less that β_0 and hence remove any questions about the fitness of $f_r(\tau)$ in (9). One could certainly obtain estimates which ensured this without such a heavy-handed modification of (8); however, a simple realization of this is presently out of authors reach.

We will work in a statistical steady state guarantied by ref. 11 (regardless if it is unique). Hence all expectations will be constant in time and no dependence on the initial data remains. From Lemma 4.2, we see that

$$\mathbb{E}\tau \geqslant \nu \int_0^t \exp\{-C(t-s)[\gamma + \sqrt{\mathbb{E}h}]\}\, ds \geqslant \frac{\nu}{\gamma + \sqrt{\mathbb{E}h}}$$

Next we use the estimate of $\mathbb{E}h$ from Proposition 4.3 and the facts that $\gamma > C\,\frac{\nu}{\beta_1}$ and $\beta_1 < \beta_2$ to obtain

$$\mathbb{E}\tau \geqslant \frac{\nu}{C_1\,\frac{\nu}{\beta_1} + \sqrt{C_2\,\frac{f_r^*}{\nu} + C_3\,\frac{\mathbb{E}\,|A^r\mathbf{u}|_{\mathbb{L}^2}^{\frac{3}{2}}}{\nu}}}.$$

To get an order of magnitude estimate we fix $2\beta_1 = \beta_0$ and recall that $f_r^* = O(C_0)$ to obtain (with new constants)

$$\mathbb{E}\tau \geqslant C\,\frac{\nu^{\frac{3}{2}}}{C_1\,\frac{\nu^{\frac{3}{2}}}{\beta_0} + \sqrt{C_0 + C_2\mathbb{E}\,|A^r\mathbf{u}|_{\mathbb{L}^2}^{\frac{3}{2}}}}. \tag{14}$$

Note that in all of the above estimates the constants are not dimensionless as the dependence on the domain still remains.

Of course, if one is interested in the behavior of these estimates as $\nu \to 0$ with a fixed forcing, one needs to compensate for the growth of $\mathbb{E}\,|A^r\mathbf{u}|^3_{\mathbb{L}^2}$ as $\nu \to 0$. For fixed $\boldsymbol{\sigma}_{\mathbf{k}}$'s as $\nu \to 0$,

$$\mathbb{E}\tau \geqslant C\frac{\nu^{\frac{3}{2}}}{\sqrt{\mathbb{E}\,|A^r\mathbf{u}|^3_{\mathbb{L}^2}}}.$$

Using the rough estimates of the growth of the $\mathbb{E}\,|A^r\mathbf{u}|^3_{\mathbb{L}^2}$ from Section 4 in the appendix one obtains $\mathbb{E}\tau \geqslant C\nu^{\frac{25}{2}}$ as $\nu \to 0$. This is known to not be optimal. The rate obtained in ref. 7 is much better. Our reliance on $|A^r\mathbf{u}|^3_{\mathbb{L}^2}$ with $r > \frac{5}{2}$ is likely the source of the problem . It is more difficult to get control of these norms that for $r \leqslant 2$ where the energy and enstrophy estimates give better scaling estimates. However the point here was to obtain estimates which provided easily managed control over the fluctuations of the the constants. Equations (10) and (9) realize this global.

8. CONCLUSION

There are a number of methods which have been used to prove analyticity of the 2D Navier–Stokes equations and related equations. The are not all equal. This shows how one specific method from the recent literature is well-suited to the forced case where one is interested in the long term behavior of the process. The formulation here leads to estimates which are easier to control than those in ref. 6. However the estimates still suffer from the deficiency which drove the author to produce three different proofs in ref. 6. Namely the decay rate τ is not asymptotically constant. It is surprising that a physical decay rate is random. This is likely an artifact of the proof but its resilience to a number of different attacks makes it an interesting open question. In fact, it was an attempt to answer this question with a direct hands on calculation which lead to ref. 18.

APPENDIX A: A PROBABILISTIC ESTIMATE

Lemma A.1. Let $M(s)$ be a continuous martingale with quadratic variation $[M, M](s)$ such that $\mathbb{E}[M, M] < \infty$. Define the semi-martingale $N(s) = -\frac{\alpha}{2}[M, M](s) + M(s)$ for any $\alpha > 0$. If $\gamma \geqslant 0$ then for any $\beta \geqslant 0$ and $T > \frac{1}{\beta}$ we have

$$\mathbb{P}\left\{\sup_{t\in[T-\frac{1}{\beta},T]}\int_0^t e^{-\gamma(t-s)}\,dN(s) > \frac{e^{\frac{\gamma}{\beta}}}{\alpha}K\right\} < e^{-K}$$

and

$$\mathbb{P}\left\{\int_0^t e^{-\gamma(t-s)}\, dN(s) < \frac{e^{\frac{\gamma}{\beta}}}{\alpha}\,[K + a\log(\beta t+1)] \text{ for all } t \geqslant 0\right\} \geqslant 1 - e^{-K}\zeta(a)$$

where $\zeta(a) = \sum_{k=1}^{\infty} k^{-a}$. (Where if $\gamma = 0$ and $\beta = 0$ then $\gamma/\beta = 0$.)

Proof. We begin with the case $\beta > 0$. Setting $b(t) = \frac{e^{\frac{\gamma}{\beta}}}{\alpha}\,[C + a\log(\beta t+1)]$ we have

$$\begin{aligned}\mathbb{P}&\left\{\int_0^t e^{-\gamma(t-s)}\, dN(s) < b(t) \text{ for all } t \geqslant 0\right\}\\ &\geqslant 1 - \sum_{n=1}^{\infty} \mathbb{P}\left\{\sup_{t\in[\frac{n-1}{\beta},\frac{n}{\beta}]} \int_0^t e^{-\gamma(t-s)}\, dN(s) \geqslant b\left(\frac{n-1}{\beta}\right)\right\}.\end{aligned}$$

We estimate the terms in this summation by

$$\begin{aligned}\mathbb{P}&\left\{\sup_{t\in[\frac{n-1}{\beta},\frac{n}{\beta}]} e^{-\gamma t}\int_0^t e^{\gamma s}\, dN(s) \geqslant b(n)\right\}\\ &\leqslant \mathbb{P}\left\{\sup_{t\in[\frac{n-1}{\beta},\frac{n}{\beta}]} \int_0^t e^{\gamma s}\, dN(s) \geqslant b\left(\frac{n-1}{\beta}\right) e^{\frac{\gamma}{\beta}(n-1)}\right\}\\ &\leqslant \mathbb{P}\left\{\sup_{t\in[\frac{n-1}{\beta},\frac{n}{\beta}]} \int_0^t -\frac{\alpha e^{-\gamma s}}{2}\, e^{2\gamma s}\, d[M,M](s) + e^{\gamma s}\, dM(s) \geqslant b\left(\frac{n-1}{\beta}\right) e^{\frac{\gamma}{\beta}(n-1)}\right\}\\ &\leqslant \mathbb{P}\left\{\sup_{t\in[\frac{n-1}{\beta},\frac{n}{\beta}]} \int_0^t -\frac{\alpha e^{-\frac{\gamma}{\beta}n}}{2}\, e^{2\gamma s} d[M,M](s) + e^{\gamma s}\, dM(s) \geqslant b\left(\frac{n-1}{\beta}\right) e^{\frac{\gamma}{\beta}(n-1)}\right\}.\end{aligned}$$

Notice that the quadratic variation of $\int e^{\gamma s}\, dM(s)$ is precisely $\int e^{2\gamma s}\, d[M,M](s)$. Also recall that a standard variation of the Kolmogorov–Doob martingale inequality implies that for a continuous L^2 martingale $\tilde{M}$, $\mathbb{P}\{\sup_{[0,t]} -\frac{A}{2}[\tilde{M},\tilde{M}] + \tilde{M} > B\} \leqslant \exp(-AB)$ (see refs. 19 and 20). Combining these observations produces

$$\begin{aligned}\mathbb{P}&\left\{\sup_{t\in[\frac{n-1}{\beta},\frac{n}{\beta}]} e^{-\gamma t}\int_0^t e^{\gamma s}\, dN(s) \geqslant b\left(\frac{n-1}{\beta}\right)\right\}\\ &\leqslant \exp\left(-\alpha e^{-\frac{\gamma}{\beta}n} b\left(\frac{n-1}{\beta}\right) e^{\frac{\gamma}{\beta}(n-1)}\right) = \frac{e^{-C}}{n^a}.\end{aligned}$$

Taking $a=0$ gives the first quoted result. Summing this estimate produces the second quoted result (recall that $\zeta(a)=\sum_{k=1}^{\infty} k^{-a}$).

When $\beta=0$, things are simpler. We need only consider the case when $\gamma=0$, otherwise the bound is trivially satisfied. If both γ and β equal zero then the statement reduced to $\mathbb{P}\{\sup_t N(t)>\frac{1}{\alpha}K\}<e^{-K}$. This is simply the basic martingale estimate cited above. ∎

APPENDIX B: CONTROL OF ENSTROPHY

We begin by considering the stochastic version of the classical enstrophy estimates. The proofs can be found in refs. 6 and 10.

Lemma B.1. For any $p>1$,

$$\mathbb{E}\,|\Lambda \mathbf{u}(t)|_{\mathbb{L}^2}^2 \leqslant e^{-2\nu t}\mathbb{E}\,|\Lambda \mathbf{u}(0)|_{\mathbb{L}^2}^2+\frac{\mathscr{E}_1}{2\nu}(1-e^{-2\nu t})$$

$$\mathbb{E}\,|\Lambda \mathbf{u}(t)|_{\mathbb{L}^2}^{2p} \leqslant e^{-2\nu t}\mathbb{E}\,|\Lambda \mathbf{u}(0)|_{\mathbb{L}^2}^{2p}+C_1\int_0^t e^{-2\nu(t-s)}\mathbb{E}\,|\Lambda \mathbf{u}(s)|_{\mathbb{L}^2}^{2(p-1)}\,ds$$

$\mathscr{E}_1=\sum_{|k|} 4\pi^2\,|k|^2\,|\sigma_k|^2$ and

$$|\Lambda \mathbf{u}(t)|_{\mathbb{L}^2}^2=|\Lambda \mathbf{u}(t_0)|_{\mathbb{L}^2}^2+\mathscr{E}_1(t-t_0)$$
$$-2\nu\int_{t_0}^t |\Lambda \mathbf{u}(s)|_{\mathbb{L}^2}^2\,ds+2\int_{t_0}^t \langle \Lambda^2\mathbf{u}(s),d\mathbf{W}(s)\rangle_{\mathbb{L}^2}.$$

Using these lemmas and the probabilistic estimates from the last section, we explore the long time behavior. Integrating up one half of the dissipative term we obtain

$$|\Lambda \mathbf{u}(t)|_{\mathbb{L}^2}^2=|\Lambda \mathbf{u}(t_0)|_{\mathbb{L}^2}^2\,e^{-\nu(t-t_0)}+\frac{\mathscr{E}_1}{\nu}(1-e^{-\nu(t-t_0)})+F(t) \tag{15}$$

where $F(t)=-\nu\int_{t_0}^t e^{-\nu(t-s)}\,|\Lambda \mathbf{u}(s)|_{\mathbb{L}^2}^2\,ds+2\int_{t_0}^t e^{-\nu(t-s)}\langle \Lambda^2\mathbf{u}(s),d\mathbf{W}(s)\rangle_{\mathbb{L}^2}$. Setting $M(t)=2\int_{t_0}^t \langle \Lambda^2\mathbf{u}(s),d\mathbf{W}(s)\rangle_{\mathbb{L}^2}$, we observe that if $[M,M](t)$ denotes the quadratic variation of $M(t)$ then

$$[M,M](t)=4\int_0^t \sum |\mathbf{k}|^4\,|\boldsymbol{\sigma}_{\mathbf{k}}|^2\,|\mathbf{u}_{\mathbf{k}}(s)|^2\,ds$$
$$\leqslant 4\hat{\sigma}(1)^2\int_0^t \sum |\mathbf{k}|^2\,|\mathbf{u}_{\mathbf{k}}(s)|^2\,ds=4\hat{\sigma}(1)^2\int_0^t |\Lambda \mathbf{u}(s)|_{\mathbb{L}^2}^2$$

where $\hat{\sigma}(n) = \sqrt{\max |\mathbf{k}|^{2n} |\boldsymbol{\sigma}_\mathbf{k}|^2}$. (Given our standing assumption on the $\boldsymbol{\sigma}_\mathbf{k}$ this is finite for all n.) Continuing, we have

$$-\nu \int_{t_0}^{t} |\Lambda \mathbf{u}(s)|_{\mathbb{L}^2}^2 \, ds + M(t) \leqslant -\frac{1}{2} \frac{\nu}{2\hat{\sigma}(1)^2} [M, M](t) + M(t).$$

Combining this estimate with Lemma A.1 produces

$$\mathbb{P}\left\{ \sup_{t \in [T-\frac{1}{\nu}, T]} F(t) > \frac{2 |\hat{\sigma}(1)|^2 e^1}{\nu} K \right\} < e^{-K}$$

and

$$\mathbb{P}\left\{ F(t) < \frac{2 |\hat{\sigma}(1)|^2 e^1}{\nu} [K + 2 \log(1+\nu t)] \text{ for all } t \geqslant 0 \right\} > 1 - \frac{\pi^2}{6} e^{-K}.$$

Using these estimates produces,

Lemma B.2. With at least probability $1 - \frac{\pi^2}{6} e^{-K}$, $|\Lambda \mathbf{u}(t)|_{\mathbb{L}^2}^2 \leqslant g_1(t, K)$ for all $t \geqslant 0$ where

$$g_1(t, K) = |\Lambda \mathbf{u}(0)|_{\mathbb{L}^2}^2 e^{-\nu t} + \frac{\mathscr{E}_1}{\nu} (1 - e^{-\nu t}) + \frac{2\hat{\sigma}(1)^2 e^1}{\nu} [K + 2 \log (1 + \nu t)]$$

for all $t \geqslant 0$. We also have

$$\mathbb{P}\left\{ \sup_{t \in [T-\frac{1}{\nu}, T]} |\Lambda \mathbf{u}(t)|_{\mathbb{L}^2}^2 - |\Lambda \mathbf{u}(0)|_{\mathbb{L}^2}^2 e^{-\nu t} > \frac{\mathscr{E}_1}{\nu} + \frac{2 |\hat{\sigma}(1)|^2 e^1}{\nu} K \right\} < e^{-K}.$$

This lemma implies the following corollary.

Corollary B.3. If $|\Lambda \mathbf{u}(0)|_{\mathbb{L}^2}^2 < \infty$ then $\mathbb{E} |\Lambda \mathbf{u}(t)|_{\mathbb{L}^2}^{2p} < \infty$ for any $p > 1$ and $t > 0$.

APPENDIX C: A NONLINEAR ESTIMATE

Lemma C.1. For $r \geqslant 2$, $\epsilon > 0$, and $u \in \mathbb{H}^{r+1}$ the following holds

$$|\langle \Lambda^r \mathbf{u}, \Lambda^r B(\mathbf{u}, \mathbf{u}) \rangle_{\mathbb{L}^2}| \leqslant \frac{C}{\epsilon^{2r-1}} |\Lambda \mathbf{u}|_{\mathbb{L}^2}^{2(r+1)} + \epsilon |\Lambda^{r+1} \mathbf{u}|_{\mathbb{L}^2}^2$$

for some $C > 0$.

Proof. This result is fairly standard. It is more or less the 2D version of an estimate at the start of Chapter 4 of ref. 21 or Chapter 6 of ref. 2. We give the basic outline for completeness. Start by observing that

$$|\langle \Lambda^r \mathbf{u}, \Lambda^r B(\mathbf{u}, \mathbf{u})\rangle_{\mathbb{L}^2}| \leqslant C\, |\nabla \mathbf{u}|_\infty\, |\Lambda^r \mathbf{u}|_{\mathbb{L}^2}^2.$$

In 2D if $r \geqslant 2$, a Gagliardo–Nirenberg inequality gives

$$|\nabla \mathbf{u}|_\infty \leqslant C\, |\Lambda \mathbf{u}|_{\mathbb{L}^2}^{1-\frac{1}{r}}\, |\Lambda^{r+1} \mathbf{u}|_{\mathbb{L}^2}^{\frac{1}{r}}.$$

By interpolation $|\Lambda^r_{\mathbb{L}^2} u|_{\mathbb{L}^2} \leqslant C\, |\Lambda \mathbf{u}|_{\mathbb{L}^2}^{\frac{1}{r}}\, |\Lambda_{\mathbb{L}^2}^{r+1} \mathbf{u}|_{\mathbb{L}^2}^{1-\frac{1}{r}}$. Combining these estimates gives

$$|\langle \Lambda^r \mathbf{u}, \Lambda^r B(\mathbf{u}, \mathbf{u})\rangle_{\mathbb{L}^2}| \leqslant C\, |\Lambda \mathbf{u}|_{\mathbb{L}^2}^{1+\frac{1}{r}}\, |\Lambda^{r+1} \mathbf{u}|_{\mathbb{L}^2}^{2-\frac{1}{r}}$$

Lastly, we use $ab \leqslant a^p/p + b^q/q$ with $p = \frac{2r}{2r-1}$ and $q = 2r$ to complete to lemma. ▮

APPENDIX D: CONTROL OF HIGHER SOBOLEV NORMS

We now use the nonlinear estimate from the last section to obtain estimates on the higher Sobolev. We begin by applying Itô's formula to $u \mapsto |\Lambda^r \mathbf{u}|_{\mathbb{L}^2}^2 = \sum |\mathbf{k}|^{2r}\, |\mathbf{u}_\mathbf{k}|^2$ to obtain

$$\begin{aligned} d\, |\Lambda^r \mathbf{u}(s)|_{\mathbb{L}^2}^2 = [&-2\nu\, |\Lambda^{r+1}\mathbf{u}|_{\mathbb{L}^2}^2 + 2\langle \Lambda^r \mathbf{u}, \Lambda^r B(\mathbf{u}, \mathbf{u})\rangle_{\mathbb{L}^2} + \mathscr{E}_r]\, ds \\ &+ 2\langle \Lambda^{2r} \mathbf{u}(s), d\mathbf{W}(s)\rangle_{\mathbb{L}^2} \end{aligned}$$

Recall that $\mathscr{E}_r = \sum |\mathbf{k}|^{2r}\, |\boldsymbol{\sigma}_\mathbf{k}|^2$. If we restrict to the case when $r \geqslant 2$, we can use Lemma C.1 with $\epsilon = \nu$ to control the nonlinearity. Doing so and integrating up half of the remaining dissipation produces

$$\begin{aligned} |\Lambda^r \mathbf{u}(t)|_{\mathbb{L}^2}^2 \leqslant{}& |\Lambda^r \mathbf{u}(0)|_{\mathbb{L}^2}^2\, e^{-\frac{\nu}{2}t} + \frac{2\mathscr{E}_r}{\nu}\,(1 - e^{-\frac{\nu}{2}t}) \\ &+ \frac{C}{\nu^{2r-1}} \int_0^t e^{-\frac{\nu}{2}(t-s)}\, |\Lambda \mathbf{u}(s)|_{\mathbb{L}^2}^{2(r+1)}\, ds + \int_0^t e^{-\frac{\nu}{2}(t-s)}\, dN(s) \end{aligned} \tag{16}$$

where $dN(t) = -\frac{\nu}{2}\, |\Lambda^{r+1}\mathbf{u}(t)|_{\mathbb{L}^2}^2\, dt + dM(t)$ and $dM(t) = 2\langle \Lambda^{2r}\mathbf{u}(t), d\mathbf{W}(t)\rangle_{\mathbb{L}^2}$.

We begin by establishing control on $\mathbb{E}\, |\Lambda^r \mathbf{u}(t)|_{\mathbb{L}^2}^2$. We use the same ideas used to obtain control of $\mathbb{E}h(t)$ in Section 4 or that were used in refs. 10 and 15 to control the expected value of the energy, enstrophy and various Sobolev norms. We take the expectation of (16). If we know that the expectation of the martingale term is zero then we would obtain the desired bound. However, we do not know *a priori* that the quadratic variation of this martingale has finite expectation. By introducing the stopping time $T=\inf\{t\colon |\Lambda^r u(t)|^2_{\mathbb{L}^2}\geqslant U\}$ we can define a stopped version of the process $u^T(t)=u(t\wedge T)$. For u^T it is clear that the expected value of the quadratic variation is finite. Since the obtained estimate is independent of the cutoff U, we can take the limit as $U\to\infty$ to obtain

Lemma D.1.

$$\mathbb{E}\,|\Lambda^r \mathbf{u}(t)|^2_{\mathbb{L}^2}+\int_0^t \mathbb{E}e^{-\frac{\nu}{2}(t-s)}\,|\Lambda^{r+1}\mathbf{u}(s)|^2_{\mathbb{L}^2}$$

$$\leqslant \mathbb{E}\,|\Lambda^r\mathbf{u}(0)|^2_{\mathbb{L}^2}\,e^{-\frac{\nu}{2}t}+\frac{2\mathscr{E}_r}{\nu}(1-e^{-\frac{\nu}{2}t})+\frac{C}{\nu^{2r-1}}\int_0^t e^{-\frac{\nu}{2}(t-s)}\mathbb{E}\,|\Lambda\mathbf{u}(s)|^{2(r+1)}_{\mathbb{L}^2}\,ds.$$

Next we obtain some estimates bounding the growth of typical solutions. Observe that

$$d[M,M](t)=4\sum|\mathbf{k}|^{4r}\,|\boldsymbol{\sigma}_{\mathbf{k}}|^2\,|\mathbf{u}_{\mathbf{k}}|^2\,dt$$

$$\leqslant 4\hat{\sigma}(r-1)\sum|\mathbf{k}|^{2r}\,|\mathbf{u}_{\mathbf{k}}|^2\,dt=4\hat{\sigma}(r-1)\,|\Lambda^r\mathbf{u}|^2_{\mathbb{L}^2}\,dt.$$

Hence

$$N(t)\leqslant -\frac{1}{2}\frac{\nu}{\hat{\sigma}(r-1)}[M,M](t)+M(t)$$

so

$$\mathbb{P}\left\{\int_0^t e^{\frac{\nu}{2}(t-s)}\,dN(s)\leqslant\frac{e^1\hat{\sigma}(r-1)}{\nu}\left[K+2\log\left(\frac{\nu 2}{t}+1\right)\right]\text{for all } t\geqslant 0\right\}\geqslant 1-\frac{\pi^2}{6}e^{-K}$$

and we have

Lemma D.2. With probability at least $1-\frac{\pi^2}{6}e^{-K}$

$$|\Lambda^r\mathbf{u}(t)|^2_{\mathbb{L}^2}-|\Lambda^r\mathbf{u}(0)|^2_{\mathbb{L}^2}\,e^{-\frac{\nu}{2}t}+\int_0^t e^{-\frac{\nu}{2}(t-s)}\,|\Lambda^{r+1}\mathbf{u}(s)|^2_{\mathbb{L}^2}\,ds$$

$$\leqslant\frac{2\mathscr{E}_r}{\nu}(1-e^{-\frac{\nu}{2}t})+\frac{C}{\nu^{2r-1}}\int_0^t e^{-\frac{\nu}{2}(t-s)}\,|\Lambda\mathbf{u}(s)|^{2(r+1)}_{\mathbb{L}^2}\,ds$$

$$+\frac{e^1\hat{\sigma}(r-1)}{\nu}\left[K+2\log\left(\frac{\nu 2}{t}+1\right)\right]$$

for all $t\geqslant 0$.

Combining this lemma with Lemma B.2, produced

Corollary D.3. With at least probability $1-\frac{\pi^2}{3}e^{-K}$,

$$|\Lambda^r \mathbf{u}(t)|_{\mathbb{L}^2}^2+\int_0^t e^{-\frac{\nu}{2}(t-s)}\,|\Lambda^{r+1}\mathbf{u}(s)|_{\mathbb{L}^2}^2\,ds\leqslant g_r(t,K)$$

for all $t\geqslant 0$ where

$$g_r(t,K)=|\Lambda^r\mathbf{u}(0)|_{\mathbb{L}^2}^2\,e^{-\frac{\nu}{2}t}+\frac{2\mathscr{E}_r}{\nu}\,(1-e^{-\frac{\nu}{2}t})+\frac{2C}{\nu^{3r-1}}\,(1-e^{-\frac{\nu}{2}(t-s)})\,g_1(t,K)^{r+1}$$
$$+\frac{e^1\hat{\sigma}(r-1)}{\nu}\left[K+2\log\left(\frac{\nu 2}{t}+1\right)\right].$$

ACKNOWLEDGMENTS

The author acknowledges stimulating conversations with David Levermore, Fabrice Plancon, and Yakov Sinai. In particular, it was Professor Levermore's suggestion to look at ref. 5, after seeing the authors earlier proofs, which led to this paper.

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